Superfluidity. Liquid helium.

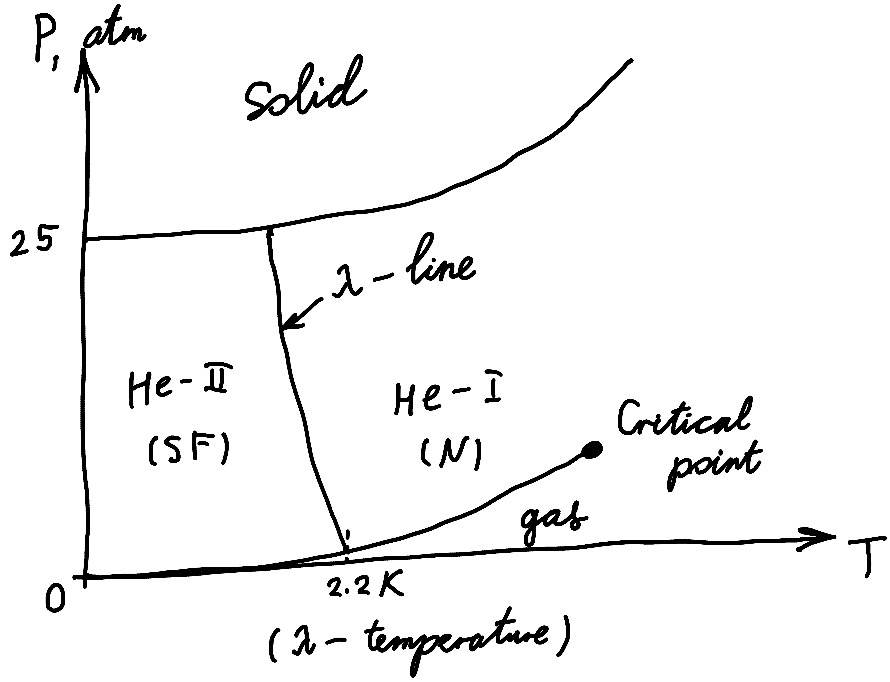
Superfluidity = the ability of a liquid to flow without viscosity (without dissipation)

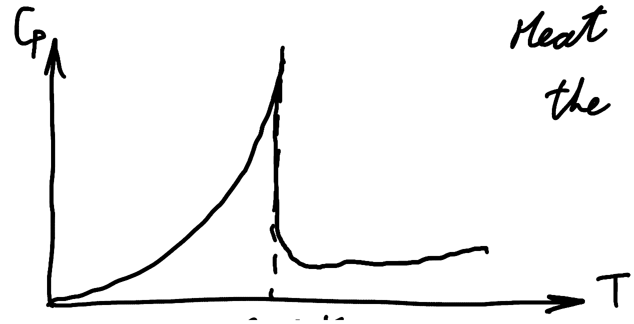
He^3, He^4, some ultracold atomic gases (e.g. ^{87}Rb, Li^6)

— Flow through a capillary without friction

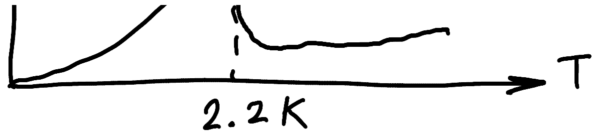
— The existence of vortices which never decay

In many aspects, superfluidity is similar to superconductivity; He^4 Bose-condenses below a certain temperature; He^3 are fermions but may form smth. similar to Cooper pairs.





Heat capacity across the λ-point



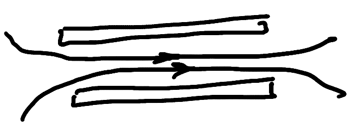
(Does not crystallise up to $T=0$!)

A strongly interacting system → a gas of weakly interacting excitations

System = the ground state + elementary excitations

How does superfluidity become possible?

Consider liquid helium flowing through a capillary with velocity $\vec{v}_s$



$\varepsilon(\vec{p})$ – excitation energy

Transformation of energy when changing a ref. frame

$$i\partial_t \Psi(\vec{r},t) = \hat{H}\,\Psi(\vec{r},t)$$

$\Psi(\vec{r},t)$ – the w.f. in a moving ref. frame

$\varphi = \Psi(\vec{r}-\vec{v}t, t)$ – the w.f. in the lab. frame

$$i\dot{\varphi} = -i\vec{v}\,\partial_{\vec{r}}\varphi + i\partial_t\varphi = \vec{v}\hat{\vec{p}}\,\varphi + i\partial_t\varphi = (\vec{v}\hat{\vec{p}} + \hat{H})\varphi$$

Thus, $\hat{H}' = \hat{H} + \vec{v}\hat{\vec{p}}$

Then the energy of an excitation in the lab. frame is $\vec{v}_s\vec{p} + \varepsilon(\vec{p})$

The total energy of the liquid

$$E = E_{kin} + \varepsilon(\vec{p}) + \vec{p}\cdot\vec{v}_s$$

dissipation) if

the total energy

$$E = E_{kin} + \varepsilon(\vec{p}) + \vec{p}\cdot\vec{v}_s$$

Excitation are created (= dissipation) if $\varepsilon(\vec{p}) + \vec{p}\cdot\vec{v}_s < 0$, i.e. if $\varepsilon(\vec{p}) - p v_s < 0$, i.e. if

$$v_s > \frac{\varepsilon(p)}{p}$$

If $\frac{\varepsilon(p)}{p}$ has a minimum, then the flow velocity has to exceed a critical value

$$\boxed{v_s = \min \frac{\varepsilon(p)}{p}}$$

in order to start dissipating energy.

Excitations in liquid helium

Long-wave

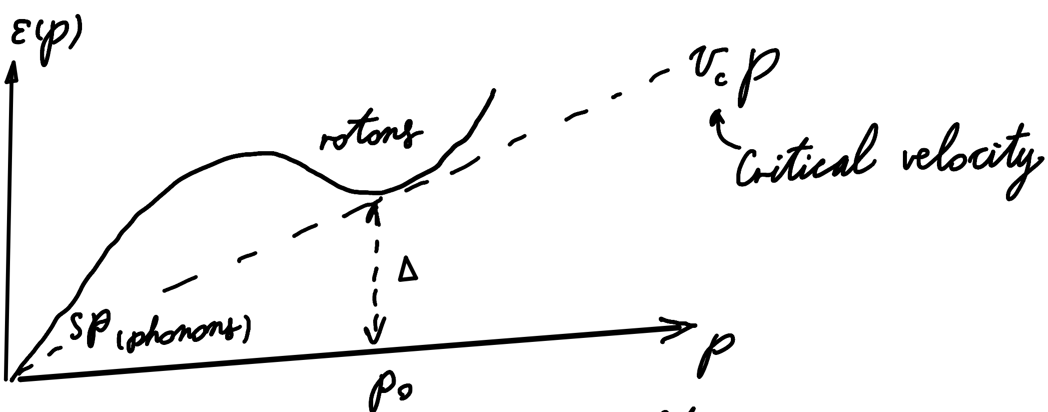
$\varepsilon_p = sp$

similar to phonons (sound waves)

short-wavelength

$\varepsilon_{\vec{p}} = \Delta + \frac{(p-p_0)^2}{2m}$

(rotons)



Conditions necessary for superfluidity:

- the existence of a critical velocity
- BEC

– BEC

Thermodynamics of superfluid helium

$\Omega = \Omega_{short} + \Omega_{long}$

$\mu = 0$, because excitations may be created out of nowhere

$(\hbar = 1)$

$$\Omega_{short} = VT\int\frac{d\vec{p}}{(2\pi)^3}\ln\left(1-e^{-\frac{\varepsilon_{\vec{p}}}{T}}\right) =$$

$$= VT\int_0^\infty\frac{4\pi p^2 dp}{(2\pi)^3}\ln\left(1-e^{-\frac{ps}{T}}\right) =$$

$$= \frac{VT^4}{2\pi^2 s^3}\int_0^\infty z^2\ln(1-e^{-z})\,dz =$$

$$= \underbrace{\frac{VT^4}{2\pi^2 s^3}\frac{z^3}{3}\ln(1-e^{-z})\Big|_0^\infty}_{0} - \frac{VT^4}{6\pi^2 s^3}\underbrace{\int\frac{z^3\,dz}{e^z-1}}_{\frac{\pi^4}{15}} = -\frac{\pi^2 VT^4}{90 s^3}$$

$$\Omega_{short} = -\frac{\pi^2 VT^4}{90 s^3\hbar^3} \quad \text{(recovered } \hbar\text{)}$$

$$P_{sh} = \frac{\pi^2 T^4}{90 s^3} \quad \text{– pressure}$$

$$S_{sh} = \frac{2\pi^2 VT^3}{45 s^3} \quad \text{– entropy}$$

$$C_{sh} = \frac{2\pi^2 V}{15 s^3}T^3 \quad \text{– heat capacity}$$

The long-wave contributions

Focus on the limit $\Delta \gg T$, which is realistic

$$\Omega_{\ldots} = -VT\int\frac{d\vec{p}}{\ldots}\ln\left(1-e^{-\frac{\varepsilon_{\vec{p}}}{T}}\right) \approx -VT\int\frac{d\vec{p}}{(2\pi)^3}e^{-\frac{\varepsilon_{\vec{p}}}{T}} =$$

Focus on the

$$\Omega_{long} = VT\int \frac{d\vec{p}}{(2\pi)^3} \ln\left(1 - e^{-\frac{\varepsilon_{\vec p}}{T}}\right) \approx -VT\int \frac{d\vec{p}}{(2\pi)^3} e^{-\frac{\varepsilon_{\vec p}}{T}} =$$

$$= -\frac{VT}{2\pi^2} e^{-\frac{\Delta}{T}} \int_0^\infty e^{-\frac{(p-p_0)^2}{2mT}} p^2\, dp \approx$$

Extend to $-\infty$ to convert to a Gaussian integral

$$\approx -\frac{VT}{2\pi^2}(2mT)^{\frac{1}{2}} p_0^2 e^{-\frac{\Delta}{T}} \int e^{-z^2} dz$$

$$= -\frac{VT^{\frac{3}{2}}(2\pi m)^{\frac{1}{2}}}{2\pi^2} p_0^2 e^{-\frac{\Delta}{T}}$$

$$P = \frac{(2\pi m)^{\frac{1}{2}} p_0^2}{2\pi^2} T^{\frac{3}{2}} e^{-\frac{\Delta}{T}}$$

$$S \cong \sqrt{\frac{m}{2\pi^3}} \frac{p_0^2 \Delta}{T^{\frac{1}{2}}} V e^{-\frac{\Delta}{T}}$$ (differentiated only the exponential)

$$C \simeq \sqrt{\frac{m}{2\pi^3}} \frac{p_0^2 \Delta^2}{T^{\frac{3}{2}}} e^{-\frac{\Delta}{T}}$$

Two-Fluid description

superfluid helium

Condensate

Elementary excitations phonons + rotons

The total density $\rho = \rho_s + \rho_n$

ρ_s goes from 0 to ρ as T is lowered from T_c to 0

The total momentum density $\vec{j} = \rho_s \vec{v}_s + \rho_n \vec{v}_n$

Consider the reference frame where the superfluid component is at rest. Introduce $\vec{w} = \vec{v}_n - \vec{v}_s$, the ... relative to the

... $\vec{w} = v_n - v_s$, the
component is at rest. Introduce $\vec{w} = v_n - v_s$, the velocity of the normal component relative to the superfluid.

The contribution of the normal component to momentum density:

$$\rho_n \vec{w} = \int \frac{d\vec{p}}{(2\pi)^3}\, \vec{p}\, f(\varepsilon_{\vec{p}} - \vec{p}\cdot\vec{w}) \qquad (1)$$

The energy of an excitation in the n-component's ref. frame

$$f(\varepsilon_{\vec{p}} - \vec{p}\cdot\vec{w}) \approx f(\varepsilon_{\vec{p}}) - \vec{p}\cdot\vec{w}\,\frac{\partial f}{\partial \varepsilon_{\vec{p}}} + \dots \quad \text{at small } w$$

Multiplying (1) by $\vec{w}$

$$\rho_n = -\frac{1}{w^2}\int \frac{d\vec{p}}{(2\pi)^3} (\vec{p}\cdot\vec{w})^2 \frac{\partial f}{\partial \varepsilon_p} = -\frac{1}{3}\int \frac{d\vec{p}}{(2\pi)^3}\, p^2 \frac{\partial f}{\partial \varepsilon_p} =$$

$$= \frac{1}{3T}\int \frac{d\vec{p}}{(2\pi)^3} \frac{p^2 e^{\frac{\varepsilon_p}{T}}}{\left(e^{\frac{\varepsilon_p}{T}} - 1\right)^2} \qquad (\text{at } \vec{w} \to 0)$$

For the phonon contribution $\varepsilon_{\vec{p}} = ps$

$$\rho_{short} = \frac{2\pi^2 T^4}{45\, s^5} \qquad (\vec{w} \to 0)$$

Using (1), one may repeat the derivation for a finite w

$$\boxed{\rho_{short}(w) = \frac{2\pi^2 T^4}{45\, c^5}\left(1 - \frac{w^2}{s^2}\right)^{-3}}$$

The limit $w < s$ sets in.

The presence of rotons only decreases the limit

$$\rho_r \approx \frac{(2\pi m)^{\frac{1}{2}} p_0^4}{\dots^2}\,\frac{e^{-\frac{\Delta}{T}}}{\sqrt{T}}$$

$$\rho_{long} \approx \frac{(2\pi m)^{\frac{1}{2}} p_0^4}{6\pi^2} \frac{e^{-\frac{\Delta}{T}}}{\sqrt{T}}$$

total density

$$\rho_{short}(T_c) + \rho_{long}(T_c) = \rho$$

- the condition on the critical temperature T_c of the transition between a superfluid and a normal fluid, because the superfluid component $= 0$ at the transition, and $\rho_n = \rho_{short} + \rho_{long}$